Subpixel Semantic Flow Supplementary Material

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In this supplementary material we provide the details of optimizing the objective function of the proposed subpixel semantic flow approach.

1 Optimization

We first derive the minimization of the subpixel semantic flow approach without bidirectional flow consistency. Consider the objective function in the continuous spatial domain:

$$E(u,v) = \int \psi \left(\sum_k |C_2(\mathbf{p}+\mathbf{w}(\mathbf{p}),k) - C_1(\mathbf{p},k)|^2 \right) d\mathbf{p} + \alpha \int \psi \left(|\nabla u(\mathbf{p})|^2 + |\nabla v(\mathbf{p})|^2 \right) d\mathbf{p} \tag{1}$$

where $\mathbf{w}(\mathbf{p}) = (u(\mathbf{p}), v(\mathbf{p}))$ is the flow field we wish to obtain relating images I_1 and I_2, $\psi(x^2) = \sqrt{x^2+\varepsilon^2}$ with $\varepsilon = 0.001$ is a robust function, namely a differentiable, convex approximation of the L^1 norm, and $C_1(.)$ and $C_2(.)$ are densely extracted, normalized (zero mean, unit variance) Geometric Blur [1] descriptors treated as multi-channel images indexed by k. We take a gradient descent approach to minimize the above objective function and follow the notation used in [2]. Let us assume an initial guess of the flow field, $\mathbf{w_0}$, is available and we are interested in the best increment direction $d\mathbf{w} = (du, dv)$. The perturbation around the initial flow field, $\mathbf{w_0}$, yields the following objective function:

$$\begin{aligned} E(du,dv) = &\int \psi \left(\sum_k |C_2(\mathbf{p}+\mathbf{w_0}(\mathbf{p})+d\mathbf{w}(\mathbf{p}),k) - C_1(\mathbf{p},k)|^2 \right) d\mathbf{p} + \\ &\alpha \int \psi \left(|\nabla(u_0(\mathbf{p})+du(\mathbf{p}))|^2 + |\nabla(v_0(\mathbf{p})+dv(\mathbf{p}))|^2 \right) d\mathbf{p}. \end{aligned} \tag{2}$$

We linearize the correlation transform images around the initial flow field and obtain:

$$\begin{aligned} C_2(\mathbf{p}+\mathbf{w_0}(\mathbf{p})+d\mathbf{w}(\mathbf{p}),k) \approx C_2(\mathbf{p}+\mathbf{w_0}(\mathbf{p}),k) + \frac{\partial C_2(\mathbf{p}+\mathbf{w_0}(\mathbf{p}),k)}{\partial x} du(\mathbf{p}) + \\ \frac{\partial C_2(\mathbf{p}+\mathbf{w_0}(\mathbf{p}),k)}{\partial y} dv(\mathbf{p}) \end{aligned} \tag{3}$$

$$C_2(\mathbf{p}+\mathbf{w_0}(\mathbf{p})+d\mathbf{w}(\mathbf{p}),k)-C_1(\mathbf{p},k) \approx C_2(\mathbf{p}+\mathbf{w_0}(\mathbf{p}),k)-C_1(\mathbf{p},k)+ \frac{\partial C_2(\mathbf{p}+\mathbf{w_0}(\mathbf{p}),k)}{\partial x}du(\mathbf{p})+ \frac{\partial C_2(\mathbf{p}+\mathbf{w_0}(\mathbf{p}),k)}{\partial y}dv(\mathbf{p}). \quad (4)$$

We denote

$$C_2(\mathbf{p}+\mathbf{w_0}(\mathbf{p}),k)-C_1(\mathbf{p},k)=C_t(\mathbf{p},k) \quad (5)$$

$$\frac{\partial C_2(\mathbf{p}+\mathbf{w_0}(\mathbf{p}),k)}{\partial x}=C_x(\mathbf{p},k) \quad (6)$$

$$\frac{\partial C_2(\mathbf{p}+\mathbf{w_0}(\mathbf{p}),k)}{\partial y}=C_y(\mathbf{p},k) \quad (7)$$

so that

$$C_2(\mathbf{p}+\mathbf{w_0}(\mathbf{p})+d\mathbf{w}(\mathbf{p}),k)-C_1(\mathbf{p},k) \approx C_t(\mathbf{p},k)+C_x(\mathbf{p},k)\,du(\mathbf{p})+C_y(\mathbf{p},k)\,dv(\mathbf{p}). \quad (8)$$

We vectorize u_0, v_0, du, dv into U, V, dU, dV, obtain diagonal matrices $\mathbf{C}_x(k)$, and $\mathbf{C}_y(k)$ from $C_x(\mathbf{p},k)$ and $C_y(\mathbf{p},k)$ and similarly column vector $C_t(k)$ from $C_t(\mathbf{p},k)$ by spanning all pixels $\mathbf{p}$. We denote $\mathbf{D}_x$ and $\mathbf{D}_y$ to be derivative filters in the direction of x and y respectively and introduce $\delta_\mathbf{p}$ as an indicator column vector whose value is 1 only at $\mathbf{p}$. The objective function can then be evaluated on a discrete spatial domain as follows:

$$E(dU,dV)=\sum_\mathbf{p}\psi\left(\sum_k\left[C_t(\mathbf{p},k)+C_x(\mathbf{p},k)\,du(\mathbf{p})+C_y(\mathbf{p},k)\,dv(\mathbf{p})\right]^2\right)+ \alpha\sum_\mathbf{p}\psi\left(\left[\delta_\mathbf{p}^T\mathbf{D}_x(U+dU)\right]^2+\left[\delta_\mathbf{p}^T\mathbf{D}_y(U+dU)\right]^2+\left[\delta_\mathbf{p}^T\mathbf{D}_x(V+dV)\right]^2+\left[\delta_\mathbf{p}^T\mathbf{D}_y(V+dV)\right]^2\right) \quad (9)$$

$$E(dU,dV)=\sum_\mathbf{p}\psi\left(\sum_k\left[\delta_\mathbf{p}^T C_t(k)+\delta_\mathbf{p}^T\mathbf{C}_x(k)\,dU+\delta_\mathbf{p}^T\mathbf{C}_y(k)\,dV\right]^2\right)+ \alpha\sum_\mathbf{p}\psi\left(\left[\delta_\mathbf{p}^T\mathbf{D}_x(U+dU)\right]^2+\left[\delta_\mathbf{p}^T\mathbf{D}_y(U+dU)\right]^2+\left[\delta_\mathbf{p}^T\mathbf{D}_x(V+dV)\right]^2+\left[\delta_\mathbf{p}^T\mathbf{D}_y(V+dV)\right]^2\right) \quad (10)$$

Note that this discretization is due to discrete spatial domain of images and not the discretization of the flow field, hence dU and dV are considered to be continuous variables.

Let $f_\mathbf{p}$ and $g_\mathbf{p}$ be the arguments of the robust functions:

$$f_\mathbf{p}=\sum_k\left[\delta_\mathbf{p}^T C_t(k)+\delta_\mathbf{p}^T\mathbf{C}_x(k)\,dU+\delta_\mathbf{p}^T\mathbf{C}_y(k)\,dV\right]^2 \quad (11)$$

$$g_\mathbf{p}=\left[\delta_\mathbf{p}^T\mathbf{D}_x(U+dU)\right]^2+\left[\delta_\mathbf{p}^T\mathbf{D}_y(U+dU)\right]^2+\left[\delta_\mathbf{p}^T\mathbf{D}_x(V+dV)\right]^2+\left[\delta_\mathbf{p}^T\mathbf{D}_y(V+dV)\right]^2. \quad (12)$$

We have

$$E(dU,dV)=\sum_\mathbf{p}\psi(f_\mathbf{p})+\alpha\sum_\mathbf{p}\psi(g_\mathbf{p}). \quad (13)$$

Using the 1st-order necessary condition of a local minimizer, we require:

$$\frac{\partial E(dU,dV)}{\partial dU}=0, \quad \frac{\partial E(dU,dV)}{\partial dV}=0 \tag{14}$$

Let us consider $\frac{\partial E(dU,dV)}{\partial dU}$. The derivation of $\frac{\partial E(dU,dV)}{\partial dV}$ is analogous.

$$\frac{\partial E(dU,dV)}{\partial dU}=\sum_{\mathbf{p}} \psi'(f_{\mathbf{p}})\cdot\frac{\partial f_{\mathbf{p}}}{\partial dU}+\alpha\psi'(g_{\mathbf{p}})\cdot\frac{\partial g_{\mathbf{p}}}{\partial dU} \tag{15}$$

$$\begin{aligned}\frac{\partial f_{\mathbf{p}}}{\partial dU} =&\sum_k 2\cdot\left[\delta_{\mathbf{p}}^T C_t(k)+\delta_{\mathbf{p}}^T\mathbf{C}_x(k)\,dU+\delta_{\mathbf{p}}^T\mathbf{C}_y(k)\,dV\right]\cdot\frac{\partial\left[\delta_{\mathbf{p}}^T\mathbf{C}_x(k)\,dU\right]}{\partial dU}\\ =&\sum_k 2\cdot\left[\delta_{\mathbf{p}}^T C_t(k)+\delta_{\mathbf{p}}^T\mathbf{C}_x(k)\,dU+\delta_{\mathbf{p}}^T\mathbf{C}_y(k)\,dV\right]\cdot\left[\mathbf{C}_x(k)\,\delta_{\mathbf{p}}\right]\\ =&\sum_k 2\cdot\left[C_x(k)\,\delta_{\mathbf{p}}\delta_{\mathbf{p}}^T C_t(k)+\mathbf{C}_x(k)\,\delta_{\mathbf{p}}\delta_{\mathbf{p}}^T\mathbf{C}_x(k)\,dU+\mathbf{C}_x(k)\,\delta_{\mathbf{p}}\delta_{\mathbf{p}}^T\mathbf{C}_y(k)\,dV\right]\end{aligned} \tag{16}$$

We are able to rotate the direction of multiplication in the above lines because the terms inside the bracket on the left hand side are scalars.

$$\begin{aligned}\frac{\partial g_{\mathbf{p}}}{\partial dU} =&\frac{\partial}{\partial dU}\left[\left[\delta_{\mathbf{p}}^T\mathbf{D}_xU+\delta_{\mathbf{p}}^T\mathbf{D}_x dU\right]^2+\left[\delta_{\mathbf{p}}^T\mathbf{D}_yU+\delta_{\mathbf{p}}^T\mathbf{D}_y dU\right]^2\right]\\ =&2\cdot\left[\delta_{\mathbf{p}}^T\mathbf{D}_xU+\delta_{\mathbf{p}}^T\mathbf{D}_x dU\right]\cdot\left[\mathbf{D}_x^T\delta_{\mathbf{p}}\right]+2\cdot\left[\delta_{\mathbf{p}}^T\mathbf{D}_yU+\delta_{\mathbf{p}}^T\mathbf{D}_y dU\right]\cdot\left[\mathbf{D}_y^T\delta_{\mathbf{p}}\right]\\ =&2\cdot\left[\mathbf{D}_x^T\delta_{\mathbf{p}}\delta_{\mathbf{p}}^T\mathbf{D}_xU+\mathbf{D}_x^T\delta_{\mathbf{p}}\delta_{\mathbf{p}}^T\mathbf{D}_x dU+\mathbf{D}_y^T\delta_{\mathbf{p}}\delta_{\mathbf{p}}^T\mathbf{D}_yU+\mathbf{D}_y^T\delta_{\mathbf{p}}\delta_{\mathbf{p}}^T\mathbf{D}_y dU\right]\\ =&2\cdot\left[\left(\mathbf{D}_x^T\delta_{\mathbf{p}}\delta_{\mathbf{p}}^T\mathbf{D}_x+\mathbf{D}_y^T\delta_{\mathbf{p}}\delta_{\mathbf{p}}^T\mathbf{D}_y\right)(U+dU)\right]\end{aligned} \tag{17}$$

Combining the above terms, Eq. 16 and Eq. 17, we obtain:

$$\begin{aligned}\frac{\partial E(dU,dV)}{\partial dU} =&\sum_{\mathbf{p}}\psi'(f_{\mathbf{p}})\cdot\left[\sum_k 2\cdot\left[\mathbf{C}_x(k)\,\delta_{\mathbf{p}}\delta_{\mathbf{p}}^T C_t(k)+\mathbf{C}_x(k)\,\delta_{\mathbf{p}}\delta_{\mathbf{p}}^T\mathbf{C}_x(k)\,dU+\mathbf{C}_x(k)\,\delta_{\mathbf{p}}\delta_{\mathbf{p}}^T\mathbf{C}_y(k)\,dV\right]\right]+\\ &\alpha\psi'(g_{\mathbf{p}})\cdot\left[2\cdot\left[\left(\mathbf{D}_x^T\delta_{\mathbf{p}}\delta_{\mathbf{p}}^T\mathbf{D}_x+\mathbf{D}_y^T\delta_{\mathbf{p}}\delta_{\mathbf{p}}^T\mathbf{D}_y\right)(U+dU)\right]\right]\end{aligned} \tag{18}$$

We note that $\sum_{\mathbf{p}}\delta_{\mathbf{p}}\delta_{\mathbf{p}}^T=\mathbf{I}$. Introducing the diagonal matrices associated with the vector of the derivative of the robust function values at every $f_{\mathbf{p}}$ and $g_{\mathbf{p}}$, namely Ψ_f' and Ψ_g', we note that the following equalities hold:

$$\sum_{\mathbf{p}}\psi'(f_{\mathbf{p}})\cdot\left[\sum_k 2\cdot\left[\mathbf{C}_x(k)\,\delta_{\mathbf{p}}\delta_{\mathbf{p}}^T C_t(k)\right]\right]=2\cdot\Psi_f'\cdot\left[\sum_k\mathbf{C}_x(k)\cdot C_t(k)\right] \tag{19}$$

$$\sum_{\mathbf{p}}\psi'(f_{\mathbf{p}})\cdot\left[\sum_k 2\cdot\left[\mathbf{C}_x(k)\,\delta_{\mathbf{p}}\delta_{\mathbf{p}}^T\mathbf{C}_x(k)\,dU\right]\right]=2\cdot\Psi_f'\cdot\left[\sum_k\mathbf{C}_x^2(k)\right]\cdot dU \tag{20}$$

$$\sum_{\mathbf{p}}\psi'(f_{\mathbf{p}})\cdot\left[\sum_k 2\cdot\left[\mathbf{C}_x(k)\,\delta_{\mathbf{p}}\delta_{\mathbf{p}}^T\mathbf{C}_y(k)\,dV\right]\right]=2\cdot\Psi_f'\cdot\left[\sum_k\mathbf{C}_x(k)\cdot\mathbf{C}_y(k)\right]\cdot dV \tag{21}$$

$$\sum_{\mathbf{p}}\psi'(g_{\mathbf{p}})\cdot\left[2\cdot\left[\mathbf{D}_x^T\delta_{\mathbf{p}}\delta_{\mathbf{p}}^T\mathbf{D}_x\right]\right]=2\cdot\mathbf{D}_x^T\Psi_g'\mathbf{D}_x \tag{22}$$

$$\sum_{\mathbf{p}} \psi^{'}\left(g_{\mathbf{p}}\right) \cdot \left[2 \cdot \left[\mathbf{D}_y^T \delta_{\mathbf{p}} \delta_{\mathbf{p}}^T \mathbf{D}_y\right]\right] = 2 \cdot \mathbf{D}_y^T \Psi_g^{'} \mathbf{D}_y \tag{23}$$

where we have used the fact that diagonal matrix multiplication is commutative. At a local minimum we require the gradient to vanish, hence

$$\begin{aligned} \Psi_f^{'} \cdot \left[\sum_k \mathbf{C}_x^2(k)\right] \cdot dU + \Psi_f^{'} \cdot \left[\sum_k \mathbf{C}_x(k) \cdot \mathbf{C}_y(k)\right] \cdot dV + \alpha \cdot \left[\mathbf{D}_x^T \Psi_g^{'} \mathbf{D}_x + \mathbf{D}_y^T \Psi_g^{'} \mathbf{D}_y\right] \cdot dU = \\ -\Psi_f^{'} \cdot \left[\sum_k \mathbf{C}_x(k) \cdot C_t(k)\right] - \alpha \cdot \left[\mathbf{D}_x^T \Psi_g^{'} \mathbf{D}_x + \mathbf{D}_y^T \Psi_g^{'} \mathbf{D}_y\right] \cdot U \end{aligned} \tag{24}$$

The term $\mathbf{D}_x^T \Psi_g^{'} \mathbf{D}_x + \mathbf{D}_y^T \Psi_g^{'} \mathbf{D}_y$ is called the Laplacian, $\mathbf{L}$, operator. Analogously with similar derivation we obtain the first order necessary condition for dV:

$$\begin{aligned} \Psi_f^{'} \cdot \left[\sum_k \mathbf{C}_x(k) \cdot \mathbf{C}_y(k)\right] \cdot dU + \Psi_f^{'} \cdot \left[\sum_k \mathbf{C}_y^2(k)\right] \cdot dV + \alpha \cdot \left[\mathbf{D}_x^T \Psi_g^{'} \mathbf{D}_x + \mathbf{D}_y^T \Psi_g^{'} \mathbf{D}_y\right] \cdot dV = \\ -\Psi_f^{'} \cdot \left[\sum_k \mathbf{C}_y(k) \cdot C_t(k)\right] - \alpha \cdot \left[\mathbf{D}_x^T \Psi_g^{'} \mathbf{D}_x + \mathbf{D}_y^T \Psi_g^{'} \mathbf{D}_y\right] \cdot V \end{aligned} \tag{25}$$

Combining the above equalities and using a matrix-vector form we get:

$$\begin{aligned} \begin{bmatrix} \Psi_f^{'} \cdot \left[\sum_k \mathbf{C}_x^2(k)\right] + \alpha \cdot \mathbf{L} & \Psi_f^{'} \cdot \left[\sum_k \mathbf{C}_x(k) \cdot \mathbf{C}_y(k)\right] \\ \Psi_f^{'} \cdot \left[\sum_k \mathbf{C}_x(k) \cdot \mathbf{C}_y(k)\right] & \Psi_f^{'} \cdot \left[\sum_k \mathbf{C}_y^2(k)\right] + \alpha \cdot \mathbf{L} \end{bmatrix} \cdot \begin{bmatrix} dU \\ dV \end{bmatrix} = \\ - \begin{bmatrix} \Psi_f^{'} \cdot \left[\sum_k \mathbf{C}_x(k) \cdot C_t(k)\right] + \alpha \cdot \mathbf{L} \cdot U \\ \Psi_f^{'} \cdot \left[\sum_k \mathbf{C}_y(k) \cdot C_t(k)\right] + \alpha \cdot \mathbf{L} \cdot V \end{bmatrix} \end{aligned} \tag{26}$$

We solve the above linear system using coarse to fine refining scheme on a Gaussian pyramid with downsampling rate of 0.5 summarized in Algorithm 1.

Bidirectional flow consistency: Adding bidirectional flow consistency constraint extends the objective function as follows:

$$\begin{aligned} E(u,v) = \int \psi\left(\sum_k \left|C_2(\mathbf{p}+\mathbf{w}(\mathbf{p}),k) - C_1(\mathbf{p},k)\right|^2\right) d\mathbf{p} + \alpha \int \psi\left(|\nabla u(\mathbf{p})|^2 + |\nabla v(\mathbf{p})|^2\right) d\mathbf{p} + \\ \beta \int \phi\left(|\mathbf{w}(\mathbf{p}) + \mathbf{w_c}(\mathbf{p})|^2\right) d\mathbf{p} \end{aligned} \tag{27}$$

where $\mathbf{w_c}$ denotes the flow field that is intended to be consistent with. We choose L^2 norm to measure flow consistency, *i.e.,* $\phi\left(x^2\right) = x^2$. Similar to the above derivation, the perturbation around an initial flow field, $\mathbf{w_0}$, yields the following objective function:

$$\begin{aligned} E(du,dv) = \int \psi\left(\sum_k \left|C_2(\mathbf{p}+\mathbf{w_0}(\mathbf{p})+d\mathbf{w}(\mathbf{p}),k) - C_1(\mathbf{p},k)\right|^2\right) d\mathbf{p} + \\ \alpha \int \psi\left(|\nabla(u_0(\mathbf{p})+du(\mathbf{p}))|^2 + |\nabla(v_0(\mathbf{p})+dv(\mathbf{p}))|^2\right) d\mathbf{p} + \\ \beta \int \phi\left(|\mathbf{w_0}(\mathbf{p}) + d\mathbf{w}(\mathbf{p}) + \mathbf{w_c}(\mathbf{p})|^2\right) d\mathbf{p}. \end{aligned} \tag{28}$$

Algorithm 1: Subpixel Semantic Flow

Input : I_1, I_2, α, *max_iter*
Output : $\mathbf{w}$
Initialization: Set up P level pyramids of correlation transforms of Geometric Blur [1] descriptors, C_1 and C_2.

```
for level=P:-1:1 do
    if level=P then
        w = 0
    else
        upsample w to current level resolution
    end
    for iter=1:max_iter do
        compute Ψ'_f and Ψ'_g based on the current estimate of w
        solve Eq. 26
        update w; w = w + [dU; dV]
        median filter w to eliminate outliers
    end
end
```

We shall only consider the effect of the last term. Vectorizing $u_0 + u_c$ and $v_0 + v_c$ as U_c and V_c respectively and evaluating the objective function on a discrete spatial domain as done before, we introduce the following new term:

$$c_{\mathbf{p}} = \left[\delta_{\mathbf{p}}^T \left(U_c + dU\right)\right]^2 + \left[\delta_{\mathbf{p}}^T \left(V_c + dV\right)\right]^2. \tag{29}$$

The energy function on a discrete spatial domain then takes the following form:

$$E\left(dU, dV\right) = \sum_{\mathbf{p}} \psi\left(f_{\mathbf{p}}\right) + \alpha \sum_{\mathbf{p}} \psi\left(g_{\mathbf{p}}\right) + \beta \sum_{\mathbf{p}} \phi\left(c_{\mathbf{p}}\right). \tag{30}$$

Using the 1st-order necessary condition of a local minimizer we obtain the following linear system:

$$\begin{bmatrix} \Psi'_f \cdot \left[\sum_k \mathbf{C}_x^2(k)\right] + \alpha \cdot \mathbf{L} + \beta \cdot \mathbf{I} & \Psi'_f \cdot \left[\sum_k \mathbf{C}_x(k) \cdot \mathbf{C}_y(k)\right] \\ \Psi'_f \cdot \left[\sum_k \mathbf{C}_x(k) \cdot \mathbf{C}_y(k)\right] & \Psi'_f \cdot \left[\sum_k \mathbf{C}_y^2(k)\right] + \alpha \cdot \mathbf{L} + +\beta \cdot \mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} dU \\ dV \end{bmatrix} = -\begin{bmatrix} \Psi'_f \cdot \left[\sum_k \mathbf{C}_x(k) \cdot C_t(k)\right] + \alpha \cdot \mathbf{L} \cdot U + \beta \cdot U_c \\ \Psi'_f \cdot \left[\sum_k \mathbf{C}_y(k) \cdot C_t(k)\right] + \alpha \cdot \mathbf{L} \cdot V + +\beta \cdot V_c \end{bmatrix}. \tag{31}$$

When considering a pair of images, we solve the above linear system using coarse to fine refining scheme on a Gaussian pyramid with downsampling rate of 0.5 in a coordinate descent fashion, where $\mathbf{w_c}$ is replaced by current updates of $\mathbf{w_1}$ and $\mathbf{w_2}$. This is summarized in Algorithm 2.

Algorithm 2: Bidirectionally Consistent Subpixel Semantic Flow

Input : I_1, I_2, α, β, max_iter
Output : $\mathbf{w_1}$,$\mathbf{w_2}$
Initialization: Set up P level pyramids of correlation transforms of Geometric Blur [?] descriptors, C_1 and C_2.

for *level=P:-1:1* **do**
  if *level=P* **then**
    $\mathbf{w_1} = 0$
    compute Ψ'_{1f} and Ψ'_{1g} based on the current estimate of $\mathbf{w_1}$
    solve Eq. 26
    update $\mathbf{w_1}$; $\mathbf{w_1} = \mathbf{w_1} + [dU_1; dV_1]$
    median filter $\mathbf{w_1}$ to eliminate outliers
    $\mathbf{w_2} = 0$
    compute Ψ'_{2f} and Ψ'_{2g} based on the current estimate of $\mathbf{w_2}$
    solve Eq. 26
    update $\mathbf{w_2}$; $\mathbf{w_2} = \mathbf{w_2} + [dU_2; dV_2]$
    median filter $\mathbf{w_2}$ to eliminate outliers
  else
    upsample $\mathbf{w_1}$, $\mathbf{w_2}$ to current level resolution
  end
  $\mathbf{w_1}^{(0)} = \mathbf{w_1}$, $\mathbf{w_2}^{(0)} = \mathbf{w_2}$
  for *iter=1:max_iter* **do**
    compute Ψ'_{1f} and Ψ'_{1g} based on the current estimate of $\mathbf{w_1}^{(iter-1)}$
    solve Eq. 31 by replacing $\mathbf{w_c}$ with $\mathbf{w_2}^{(iter-1)}$
    update $\mathbf{w_1}^{(iter)}$; $\mathbf{w_1}^{(iter)} = \mathbf{w_1}^{(iter-1)} + [dU_1; dV_1]$
    median filter $\mathbf{w_1}^{(iter)}$ to eliminate outliers
    compute Ψ'_{2f} and Ψ'_{2g} based on the current estimate of $\mathbf{w_2}^{(iter-1)}$
    solve Eq. 31 by replacing $\mathbf{w_c}$ with $\mathbf{w_1}^{(iter-1)}$
    update $\mathbf{w_2}^{(iter)}$; $\mathbf{w_2}^{(iter)} = \mathbf{w_2}^{(iter-1)} + [dU_2; dV_2]$
    median filter $\mathbf{w_2}^{(iter)}$ to eliminate outliers
  end
end

References

[1] Alexander C. Berg and Jitendra Malik. Geometric blur for template matching. In *2001 IEEE Computer Society Conference on Computer Vision and Pattern Recognition (CVPR 2001), with CD-ROM, 8-14 December 2001, Kauai, HI, USA*, pages 607–614, 2001.

[2] Ce Liu. *Beyond Pixels: Exploring New Representations and Applications for Motion Analysis*. PhD thesis, 2009.